Monte Carlo Multi-factor Short Rate Model

The Monte Carlo Multi-factor Short Rate Mode has been used extensively in pricing a variety of interest rate derivative securities. The model assumes that short rates at reset times are lognormally distributed.

Let *I* = [0,*T*] , where *T* > 0, be an interval of time and let D*t* = *T/N*, where *N* is a positive integer, be an accrual period. We consider a model for *N* , D*t* - period short rates.

Specifically let 0 0 = *t* < < *t* = *T N* □ , where *t i T i* = D , be an equally spaced partition of *I* . We are interested in the D*t* - period short rate at time equal to *ti* , which we denote by *ri* , for *i* = 1, ... , *N* .

The FP short rate model derives from an underlying forward rate model. In particular let *f ti* , for *i* = 1,..., *N* , denote a D*t* - period forward rate, beginning at time equal to *ti* , as seen at time equal to *t* . Also let {*W_ t I*} *ti* Î , for *i* = 1, ... , *N* , denote standard Brownian motion under a probability measure *Q* ; furthermore assume that the Brownian motions {*W_ t I*} *ti* Î and {*W_ t I*} *tj* Î , for *i*, *j* = 1,..., *N* , have instantaneous constant correlation equal to _r*ij* . We assume that the process {*f t t* } *ti* Î[0,] , for *i* = 1, ... , *N* , satisfies a stochastic differential equation (SDE) of the form

$$d f_t^i = f_t^i \left(\hat{\mu}_t^i dt + \hat{\sigma}_t^i d\hat{W}_t^i \right), \qquad t \in [0, t_i],$$

where _m*ti* and _s*ti* denote piecewise constant drift and volatility terms (i.e., constant over each subinterval, [t , t) j-1 j , for j = 1, ... ,i). We then set r f i ti = (i = 1,..., N).

Notice that ri is equal to

$$f_0^i \exp\left(\left[\tilde{\mu}_i - \frac{\tilde{\sigma}_i^2}{2}\right] + \tilde{\sigma}_i Z^i\right)$$

where f i0 is the Dt - period forward rate beginning time equal to ti , as seen at time equal to zero, Z i is a standard normal random variable, . For consistency, we re-write (2) as the equivalent expression

$$f_0^i \exp\left(\mu_i + \sigma_i W^i\right)$$

$$\text{where} \quad \mu_i = \tilde{\mu}_i - \frac{\tilde{\sigma}_i^2}{2}, \quad \sigma_i = \frac{\tilde{\sigma}_i}{\sqrt{t_i}} \text{ and } W^i = \sqrt{t_i} Z^i .$$

FP assumes that short rates are lognormally distributed, of the form (3), under the probability measure Q . The model includes as input the initial term structure of forward rates (i.e., f i 0 , for i N = 1,...,). The volatilities si , for i = 1,..., N , in (3) are also supplied as inputs. Furthermore correlations for Z i and Z i+1 (i = 1, ... , N) are input and then combined with the volatilities si (i = 1, ... , N), to construct a covariance matrix (under the measure Q) for the random vector [s s] 1W1 W NN T ,..., .

Let P(j, i) , for i = 1,..., N and j = 0, ...i , denote the price at time equal to t j of a bond with face value $1 at time equal to ti . Also let {B i N} i | = 0, ... , , where D , denote our money-market, numeraire process under Q . Then the Monte Carlo construction scales the short rate (3), for i = 1,..., N , so that all zero coupon bond prices, as seen at time equal to zero, are repriced, that is, P i E. Furthermore, *because this is a short rate model,* the martingale (no-arbitrage) condition is then *automatically* satisfied under Q (where F_i denotes the sigma algebra induced by W_i ,for i = 1,..., N).

Note that

· all options are of European exercise type,
· by the *correlation between adjacent LIBOR rates* r_i *and* r_{i+1} (i = 1, ... , N -1), we mean the correlation between the standard normal random variables Z_i and Z_{i+1} described in Section 2, and,
· by the *volatility for LIBOR rate* r_i , we mean the value for the parameter s_i

In our terminology below, the payoff for a caplet with tenor t_i (i = 1,..., N) is defined equal to

$$(r_i - X)\Delta t$$

where r_i denotes the spot rate at time equal to t_i and X denotes the strike; furthermore the payoff is received at time equal to t_{i+1} . Caplets were specified based on the following parameters :

· short rate equal to three month LIBOR (see https://finpricing.com/lib/FxForwardCurve.html),
· tenor equal to three months, one year and three years,
· initial forward rate term structure set
· constant at 7%, 7.5% and 8%, and
· linearly upward rising , initially at 7% , with .0025 increments every three months,
· strike equal to 7%, 8% and 9%,
· LIBOR rate volatility set constant at 10%, 25% and 50%, and
· adjacent LIBOR rate correlation constant in the range of 90% to 99% inclusive.

Caplets were benchmarked using Black's model, that is, with constant volatility and with fixed discounting based on the initial term structure of forward rates; FP prices were based on 10,000 Monte Carlo paths. Numerical test results showed relative differences between the benchmark and FP model

- not greater than 1% for high (50%) volatility,
- not greater than .1% for low (10%) and medium (25%)volatility.