FX Option Valuation via BGM

The interest rate diffusion refers to the Brace-Gatarek-Musiela (BGM) model that is a multi-factor log-normal model. The very brief description of BGM approach in Monte Carlo setting is

❖ Select almost periodic grid of forward dates, typically 3m apart

❖ Select arbitrary grid of diffusion dates based on reset dates of exotic security which user aims to price

❖ Postulate that simple rates between any two consecutive forward dates, as observed on any diffusion date, are distributed lognormally (as well as simple rate between observation date and first forward date greater than it)

❖ Determine parameters of the lognormal distributions of the forward rates:

  ➢ First moments (means) from no-arbitrage condition that average of the discounting factors along Monte Carlo path equals discounting factor from the initial curve.
  ➢ Diagonal second moments (volatilities) from the requirement that computed prices of calibration instruments (European style options on rates) match market prices as close as possible.
  ➢ Off--diagonal second moments (correlations), from the requirement that computed correlations between interest rates match historically observed correlations, adjusted to take into account traders' view.

The straightforward approach to compute a European vanilla call or put of maturity t_k and strike K on the exchange rate $X(t_k)$ is to average over all paths the discounted pay-off of the option, i.e., for a call:

$$C(t_k, K) = \frac{1}{N} \sum_{l=1}^{N} Num^d (t'_k)[l] \left(X(t_k)[l] - K \right)_+$$

and a similar formula for a put. If the maturity falls in between two consecutive diffusion dates, then a partial diffusion from the last diffusion date before maturity to the exact maturity date is performed.

In order to increase accuracy, we observe that, on the last step, the volatility and the interest rates do not vary anymore. Hence, the option can be computed by Black-Scholes formula. This will smooth the pay-off and fasten the convergence. For intermediate maturities, this technique also avoids partial diffusion. The price of a call is given by:

$$C(t_k, K) = \frac{1}{N} \sum_{l=1}^{N} Num^d (t'_{k-1})[l] \, BS\big(X(t_{k-1})[l], K, \delta t, \sigma_X (t_{k-1})[l], r^d (t_{k-1})[l], r^f (t_{k-1})[l] \big)_+$$

where BS is the Black-Scholes formula and the rates are:

$$r^d (t_{k-1}) = -\frac{1}{\delta t} \log DF^d (t'_{k-1}, t'_k) \qquad r^f (t_{k-1}) = -\frac{1}{\delta t} \log DF^f (t'_{k-1}, t'_k)$$

In the case where the maturity T of the option falls before t_k, then we replace δt by $T - t_{k-1}$ and t'_k by the spot date T' corresponding to the observation date T.

For knock-out options, the sum is only performed on those trajectories that do not hit the barrier(s), although we still divide it by the total number N. However, especially when computing sensitivities, a discontinuity may occur if a path reaches a point very close to the barrier without crossing it, and finishes in the money.

To avoid such problems, trajectories are weighted according to their survival probability. One can imagine each trajectory as representing a whole beam of paths, of which we only know their values at the observation dates, but that can "wander" in between these dates. Thus, some of them may be killed because they hit the barrier between observation dates. By a "Brownian bridge" technique, the proportion of those who hit the barrier for each trajectory and in each interval $[t_k, t_{k+1}]$ is computed, and, in the end, the trajectory is weighted with the proportion of paths that survived all the way through. If, at one of the observation dates, the barrier is crossed, then the weight is automatically set to 0.

As for vanilla options, a closed form is used in the last step. Not only this smoothes the pay-off, but also, the singularity occurring in reverse barrier options is taken into account with accuracy.

We mention that the risk-neutral expectation of a domestic numeraire equals the original discount factor and that, by the Bayesian rule:

$$\mathbf{E}_0\left(Num^d(t)\,DF^d(t,T)\right) = DF^d(0,T)$$

In practice, the average of Monte-Carlo samples may slightly differ from its theoretical value, and this small difference with respect to the notional amount may become relatively big with respect to an option price. Moreover the drift of Libor and Future rates has been approximated by neglecting an Ito term. It is therefore important to "adjust" the average. The most straightforward way is to multiply the sample discount factors by the ratio between the initial discount factor and the actual average. This is equivalent to a rate shift. A more accurate way, but more time consuming, in the BGM model is to multiply the <u>rates</u> by a coefficient in order to meet the theoretical expectation. For reasonable values of the rate volatility, the rate shift is widely sufficiently accurate and the second technique is not necessary (it may be used if the volatility exceeds 60%).

Assume that $Num^d(t'_k)$ has the correct average. Adjusting the short discount factor $DF(t'_k, t'_{k+1})$ ensures the expectation of $Num^d(t'_{k+1})$. Then we successively adjust the spot Libor, the stub and the series of Futures.

The FX rate $X(t_k)$ is now adjusted with respect to initial foreign discount factors:

$$\mathbf{E}^d\left(Num^d(t'_k)\, X(t_k)\right) = X(t_0)\, DF^f(t'_0, t'_k)$$

This will in particular ensure the put/call parity of vanilla FX options of maturity t_k that are computed in the "non smooth" way.

To ensure the put/call parity of options of maturity t_{k+1} computed in the "smooth" way, one has to adjust, on top of this, the drift of the foreign short rate by ensuring that:

$$\mathbf{E}^d\left(Num^d\,(t'_k)\,X(t_k)\,DF^f\,(t'_k,t'_{k+1})\right) = X(t_0)\,DF^f\,(t'_0,t'_{k+1})$$

Again, once these two adjustments have been achieved, one corrects the drift of other foreign rates (spot Libor, stub and Futures) in order to meet the equalities:

$$\mathbf{E}^d\left(Num^d\,(t'_k)\,X(t_k)\,DF^f\,(t'_k,t'_k+\delta)\right) = X(0)\,DF^f\,(0,t'_k+\delta)$$

$$\mathbf{E}^d\left(Num^d\,(t'_k)\,X(t_k)\,DF^f\,(t'_k,T_j)\right) = X(0)\,DF^f\,(0,T_j)$$

Calibration is a procedure that finds which expected spot volatility to input in order to match the market price of at-the-money FX vanilla options. It works as a "bootstrap". The time interval $[0,t_{\max}]$ is divided into subintervals $[0,\tau_1],\ldots,[\tau_{m-1},\tau_m]$ where we recall that τ_i is the maturity of the i-th calibrating option. The level of $\bar{\sigma}(t)$ is computed on each interval $[\tau_{i-1},\tau_i]$, one after the other, in accordance with the market price of the τ_i option.

By assumption, calibration dates are part of the diffusion ones. We thus may define the index k_i by $t_{k_i}=\tau_i$. Two shapes of the term structure of the expected spot volatility are proposed. In the first one, referred to as "non interpolated", $\bar{\sigma}(t_k)$ is constant on each interval and makes a jump at calibration dates.

In the second one, referred to as "interpolated", $\bar{\sigma}(t_k)$ is constant on the first interval $[0,t_{k_1-1}]$, then, once $\bar{\sigma}(t_k)$ has been calculated for $0\le k\le k_i-1$, it is linearly interpolated between k_i-1

and $k_{i+1} - 1$. It is the value of $\bar{\sigma}(t_{k_{i+1}-1})$ that is calibrated on the market price of the τ_{i+1} option. This shape provides a more continuous term structure, but also more oscillatory. In particular problems may occur for very low volatilities, as oscillations may induce negative volatilities.

A secant method (see *Numerical Recipes in C*) is used at each step to match the market prices (with some improvements to be more robust).

Interest rate calibration is based on a closed form approximation of the volatility of caplets and swaptions. It only matches the approximation to the market volatilities, so repricing calibrating options with the model will not make an exact market fit. Nevertheless, most of the time the accuracy is of the order of 1% of the option price (the error may be bigger for very long-term options). Techniques are currently under study to achieve an exact market match.

In their 1997 article, Brace, Gatarek and Musiela showed that a caplet of maturity T can be priced with a Black-Scholes formula, using a volatility parameter given by:

$$\bar{\sigma} = \sqrt{\frac{1}{T-t} \int_t^T \sigma(s,T)^2 \, ds}$$

where $\sigma(s,T)$ is the volatility of the Libor future $F(t;T,T+d)$ at date t. In fact, they prove that in the "forward-neutral" measure at date $T+d$, $F(t;T,T+d)$ has a log-normal distribution and that its logarithm has variance $\int_t^T \sigma(s,T)^2 \, ds$. In other forward-neutral probabilities, the variance is approximately equal, but not quite.

Similarly, the covariance of the logarithm of two different Libor futures $F(t;T,T+d)$ and $F(t;T',T'+d)$ is approximately equal to:

$$\int_t^T \sigma(s,T)\sigma(s,T')\rho(s,T,T')\,ds$$

where $\rho(s,T,T')$ is the instantaneous correlation at date s between the two futures (here there is no possible exact formula, because futures have different maturities).

In practice, the volatility of a caplet, the maturity of which is one of the sample dates $\tilde{T}_i$ of the volatility triangle, can be calculated by adding the squares of the values over a line of the triangle:

$$\overline{\sigma}(\tilde{T}_i) = \sqrt{\frac{1}{\tilde{T}_i}\sum_{j=0}^{i-1}\sigma(\tilde{t}_j,\tilde{T}_i)^2(\tilde{t}_{j+1}-\tilde{t}_j)}$$

For swaptions, one first approximates the CMS rate by a combination of Libor futures, with coefficients evaluated according to the initial yield curve (ref. https://finpricing.com/lib/IrBasisCurve.html). Then the variance of this combination at the option maturity is calculated using the covariances of Libor futures, which we approximate as explained above.

The whole volatility triangle is now parameterized by a double summation of Chebyshev polynomials (other parameterizations are also available):

$$\sigma(t,T) = \sum_{i,j} a_{ij}\, \Phi_i(t)\, \Phi_j(T-t)$$

where Φ_i is the Chebyshev polynomial of degree i, defined by:

$$\cos i\theta = \Phi_i(\cos\theta)$$

Coefficients a_{ij} are found to minimize a least square criterion on the volatilities of calibrating options. The minimization algorithm is Levenberg-Marquardt.